Pratte and Baines find that the jet cross section grows as $\chi^{1/3}$, as for a jet in a coaxial stream (Figs. 7 and 9 in Ref. 1). It therefore seems reasonable to assume that the vortex spacing also grows as $\chi^{1/3}$, so that

$$Y_o = Y_v \chi^{1/3} \tag{8}$$

where $Y_v$ is a universal constant. Substituting Eq. (5) into Eq. (6) and writing the result in similarity variables yields

$$dZ_o/d\chi = (K'\sigma_e^2/4\pi U_\infty d_e^2)(1/Y_v^2 \chi^{2/3}) \tag{9}$$

The data of Pratte and Baines show that $Z_o$ is a universal function of $\chi$, and consequently Eq. (9) requires that

$$K'\sigma_e^2/4\pi U_\infty d_e^2 = K \tag{10}$$

where $K$ is another universal constant. Furthermore, integrating Eq. (9) leads to the result

$$Z_o = (3K/Y_v^2)\chi^{1/3} \tag{11}$$

where the constant of integration has been arbitrarily set equal to zero. The jet trajectory predicted by this equation is verified by the data of Pratte and Baines, who measured

$$Z_o = (\text{const})\chi^{1/3}$$

Finally, it is possible to write Eq. (5) in terms of similarity variables. Using Eq. (10), the result is

$$\Gamma^* = K/Y_v \chi^{1/3} \tag{12}$$

where

$$\Gamma^* = (\Gamma/4\pi U_\infty d_e)\sigma_e \tag{13}$$

Equations (12) and (13) show that the product of normalized vortex strength $(\Gamma/4\pi U_\infty d_e)$ and $\sigma_e$ is a universal function of the similarity variable $\chi^{1/3}$.

In summary, starting from a vortex spacing based on data, the model predicts the correct jet trajectory to within an empirical constant. Indirectly, at least, this appears to verify the proposed relation between vortex strength and spacing. The results derived are expected to hold within the vortex zone, which has been found to lie[1] downstream of the value $\chi = 5$. Two empirical constants have been introduced so far, $K$ and $Y_v$, and their numerical values must be found from data. From Fig. 9 of Ref. 1,

$$Y_v = 1.45$$

From Eq. (11) and Fig. 5 of Ref. 1,

$$3K/Y_v^2 = 1.63$$

Equation (12) then yields the following expression for the correlated vortex strength:

$$\Gamma^* = (0.79)/\chi^{1/3} \tag{14}$$

### 3. Conclusions

A model to represent the vortex zone within a jet in a subsonic cross flow has been developed. This vortex zone lies downstream[1] of the correlated jet axial distance $\chi = 5$. The analysis yields an expression for the strength of the counterrotating vortices in the jet as a function of distance along the jet trajectory. A correlated form of this vortex strength also results from the analysis. The model is self-consistent, since matching the measured jet spread leads to the correct form for the jet trajectory. Direct comparison of predicted and measured vortex strengths, however, cannot be made at present, due to a lack of data.

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# Heat Transfer with Nonlinear Boundary Conditions via a Variational Principle

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## I. Introduction

In this study the authors develop a technique based on a variational principle that possesses a Hamiltonian structure, and which is different from Biot's Principle.[1] The variational principle developed here corresponds to the general problem of heat conduction with finite wave speeds. The transition to the classical situation, viz., where the speed of propagation of the thermal disturbance is infinite, is accomplished by allowing the relaxation time to approach zero.

Biot,[1] by applying variational techniques, developed a method of formulating heat-transfer problems involving nonlinear boundary conditions. Biot combined the concepts of thermal potential, dissipation function, and generalized thermal forces with the variational principle to develop a powerful technique for solving nonlinear boundary value problems.

It should be pointed out that the variational principle developed here is neither a quasi-variational formulation of the problem, as is Biot's, nor a restricted variational principle, as that of Glansdorff and Prigogine.[2] The motivation here is to develop a variational technique applicable to nonlinear heat conduction problems within the framework of the classical calculus of variations. This is accomplished by considering the problem of heat conduction with finite wave speeds, Eq. (2), and making use of the exact Lagrangian.

The main drawback to this method is that, at this time, the error of the approximate solution cannot be controlled. It should, however, be noted that, to the authors' knowledge, no method of approximate solution in the physics of irreversible phenomena exists in which the error of the approximate solution can be controlled. Examples of other methods are Biot's, Galerkin's, and the integral method.

The concepts of generalized coordinates and the method of partial integration are the basic tools of this theory. By

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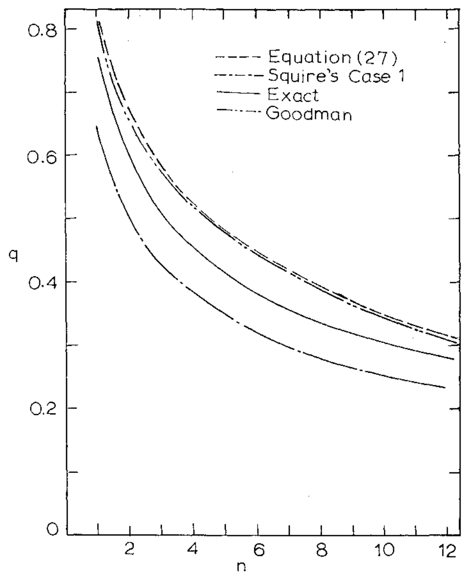


**Fig. 1 Comparison of solutions for the second case.**

means of the variational technique developed here a nonlinear boundary value problem can be reduced to a linear boundary value problem whose solution is often capable of being expressed in analytical closed form.

Following the technique of Rafalsky and Zyszkowsky,[3] boundary conditions are formulated by establishing an arbitrary balance equation on the surface of the body. This balance equation will yield a simple algebraic expression, as opposed to the differential expressions used by Biot[1] and Lardner.[4]

The fact that the solution of Eq. (2) tends to the solution of Eq. (1), as the relaxation time approaches zero, tends to physically confirm the variational principle.[5]

Finally, it is not the authors intention to solve any new problems by applying this new variational principle, but it is to point out the usefulness of this new technique in solving some well-known problems in the physics of nonlinear heat conduction.

## II. Variational Principle

A well-known property of the classical heat conduction equation, Eq. (1),

$$c\partial T/\partial T = k\nabla^2 T \tag{1}$$

is that a portion of its solution extends to infinity. This fact implies that the effects of a thermal disturbance are instantaneously manifested at a distance infinitely far from the source of the thermal disturbance.

Obviously, this feature of the classical heat conduction equation, Eq. (1), contradicts our experience. The general heat conduction equation, Eq. (2),

$$\tau c\partial^2 T/\partial t^2 + c\partial T/\partial t = k\nabla^2 T \tag{2}$$

is consistent with our experience[6,7] and the use of relation (1) can be justified by regarding it as an approximation to the general heat conduction equation.

It can be shown that the general heat conduction equation, Eq. (2), possesses a Lagrangian form.[8] This Lagrangian can be written as

$$L = \left\{\frac{c\tau}{2}\left(\frac{\partial T}{\partial t}\right)^2 - \frac{k}{2}\sum_i\left(\frac{\partial T}{\partial x_i}\right)^2\right\} e^{t/\tau} \tag{3}$$

One can obtain the general heat conduction equation, Eq. (2), by substituting the Lagrangian, Eq. (3), into the Euler-Lagrange equation

$$\frac{\partial L}{\partial T} - \frac{\partial}{\partial t} - \frac{\partial L}{\partial(\partial T/\partial t)} - \frac{\partial}{\partial x_i}\frac{\partial L}{\partial\, \partial T/\partial x_i} = 0 \tag{4}$$

A significant point is that one can obtain an approximate solution to the classical heat conduction equation, Eq. (1), by making use of the Lagrangian, Eq. (3), corresponding to the general heat conduction equation, Eq. (2), by performing the variation of the action integral

$$I = \int_t\int_V L\, dVdt \tag{5}$$

and then letting $\tau$ approach zero.

It will be tacitly assumed that no natural boundary conditions exist. This assumption is equivalent to the statement that the temperature variation $\delta T$ vanishes on the boundaries.

One advantage of the above formulation is immediately evident in that the problem can be stated in terms of Lagrangian equations in generalized coordinates which define the heat flowfield.

At this point, let us consider the physically significant problem of the one-dimensional flow of heat. The boundary condition to be applied is that $\delta T$ must vanish on the boundary.

Initially two generalized coordinates are introduced, as in the procedures of Refs. 1 and 3; the penetration distance $\delta$, and the surface temperature $q$. The penetration distance $\delta$ can be considered as an independent generalized coordinate that can be determined by the variational procedure. The generalized coordinate $q$, the surface temperature, can be determined from the following relation[3]:

$$\int_s B(s,T,t)g(s)ds = 0 \tag{6}$$

The function $B(s,T,t)$ is determined from the prescribed boundary condition

$$B(s,T,t) = 0 \tag{7}$$

where $s$ is the surface of the body and

$$g(s) \geq 0 \tag{8}$$

$g(s)$ can be interpreted as a statement expressing the accuracy desired of the approximation on the surface $s$.[3]

## III. Application of the Theory

The theory just developed is now applied to a class of problems in one space variable with fixed boundaries in which the transient heat conduction equation is involved. These problems have been solved by means of the integral method.[9]

The assumption is made that the thermal properties of the body under consideration are constant over the body. This assumption requires that the governing equation be linear. However, problems involving both linear and nonlinear boundary conditions will be considered.

To begin with let us look at the case of a semi-infinite slab whose initial temperature is zero.[10] The heat flux is a prescribed function of the surface temperature and time if we can introduce the variable

$$T(0,t) = q \tag{9}$$

the boundary condition, Eq. (7), becomes,

$$B \equiv \partial T(0,t)/\partial x + f(q,t) = 0 \tag{10}$$

Let the temperature distribution have a cubic profile of the form

$$T = \begin{matrix} q(1 - x/\delta)^3 & \text{for} \quad 0 \leq x \leq \delta \\ 0 & \text{for} \quad x > \delta \end{matrix} \tag{11}$$

where $\delta$ is the penetration distance, and both the variables $q$ and $\delta$ are unknown functions of time.

Combining the action integral, Eq. (5), with the Lagrangian, Eq. (3), yields the following action integral:

$$I = \int_{t_0}^{t_1}\int_0^{\delta}\left\{\frac{\tau}{2}\left(\frac{\partial T}{\partial t}\right)^2 - \frac{\alpha}{2}\left(\frac{\partial T}{\partial x}\right)^2\right\} e^{t/\tau}dxd\tau \qquad (12)$$

In Eq. (12) $\alpha$ is the thermal diffusivity, and the time interval $[t_0,t_1]$ is arbitrarily chosen.

By substituting the specific cubic profile, Eq. (11), into the action integral, Eq. (12), and performing the indicated integration with respect to $x$, one can obtain the relation

$$I = \int_{t_0}^{t}\left\{\frac{t}{2}\left[\frac{\delta}{7}\dot{q}^2 + \frac{1}{7}q\dot{q}\dot{\delta} + \frac{9}{105}\frac{q^2}{\delta}\dot{\delta}^2\right] - \frac{9}{10}\alpha\frac{q^2}{\delta}\right\} e^{t/\tau}dt \qquad (13)$$

or,

$$I = \int L(\dot{q},\dot{\delta},q,\delta,t)dt \qquad (14)$$

If the action integral, Eq. (13), possesses a stationary value relative to the penetration distance $\delta$, which is an independent variable, then the Euler-Lagrange equation

$$(d/dt)\partial L/\partial\dot{\delta} - \partial/\partial\delta = 0 \qquad (15)$$

must be satisfied. Substituting the Lagrangian determined in Eq. (13) into the Euler-Lagrange equation, Eq. (15), yields,

$$\frac{\tau}{2}\frac{d}{dt}\left[\frac{q\dot{q}}{7} + \frac{18q^2\dot{\delta}}{105\delta}\right] - \frac{\tau}{2}\left[\frac{\dot{q}^2}{7} - \frac{9q^2\dot{\delta}^2}{105\delta^2}\right] - \frac{1}{2}\left[\frac{q\dot{q}}{7} + \frac{18q^2\dot{\delta}}{105\delta}\right] - \frac{9\alpha q^2}{10\delta^2} = 0 \qquad (16)$$

To obtain the classical situation, let $\tau$ approach zero, and obtain

$$q\dot{q}/7 + 6q^2\dot{\delta}/35\delta = 9\alpha q^2/5\delta^2 \qquad (17)$$

The algebraic equation, Eq. (6), now assumes the form, Eq. (18), which relates the surface temperature to the penetration distance

$$\delta = 3q/f(q,t) \qquad (18)$$

Upon combining Eqs. (17) and (18) the first-order differential equation is derived

$$11q\dot{q} - (6q^2/f)(f'\dot{q} + \partial f/\partial t) = 7\alpha f^2 \qquad (19)$$

where

$$f' \equiv df/dq \qquad (20)$$

and the dot denotes differentiation with respect to the time variable. $q = 0$, when $t = 0$, is the initial condition appropriate to this problem.

Three distinct cases will be considered for which the differential equation, Eq. (19), can be analytically integrated. The three situations arise when $f$ is constant, when $f$ is a function of $t$ only and when $f$ is a function of $q$ only.

For the first case $f$ is constant, therefore Eq. (19) reduces to

$$11q\dot{q} = 7\alpha f^2 \qquad (21)$$

or

$$q = f[\tfrac{14}{11}\alpha t]^{1/2} = 1.128f[\alpha t]^{1/2} \qquad (22)$$

The same problem was solved by Goodman[1] by applying the integral method, and he found that,

$$q = 1.15g[\alpha t]^{1/2} \qquad (23)$$

The exact solution is known to be

$$q = 1.128f[\alpha t]^{1/2} \qquad (24)$$

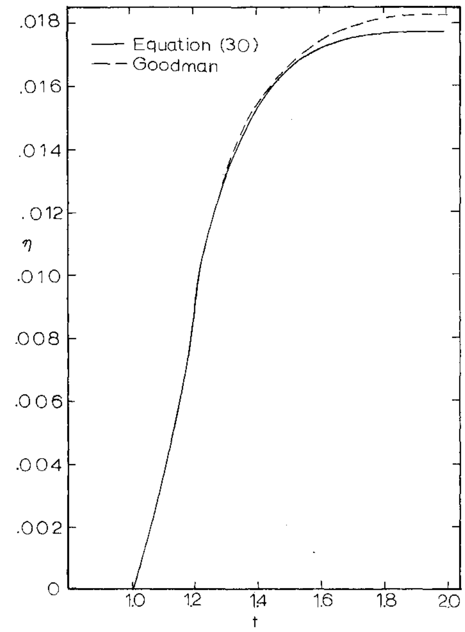


**Fig. 2 Comparison of Eq. (31) and Goodman's solution.**

and a reduction in error is evident.

In the second case $f$ is independent of $q$ and Eq. (19) assumes the form of the simple linear equation,

$$\tfrac{1}{2}d/dt(q)^2 - \tfrac{6}{11}(\dot{f}/f)q^2 = \tfrac{7}{11}\alpha f^2 \qquad (25)$$

Consider the specific case in which

$$f = t^n \qquad (26)$$

where $n \geq 1$ is an integer. The solution to Eq. (25) now becomes

$$q = [14\alpha/(10n + 11)]^{1/2}t^{n+(1/2)} \qquad (27)$$

In Fig. 1 it can be seen that this solution comes very close to Goodman's solution.

For the final case $f$ is independent of $t$, and Eq. (19) becomes

$$\int_0^q\left[\frac{11qf(q) - 6q^2f'(q)}{f^3(q)}\right]dq = 7\alpha t \qquad (28)$$

Let us assume that

$$f(q) = \frac{H}{k}(q + T_0)^4 \qquad (29)$$

This relation represents a slab of absolute zero. For the solution one obtains

$$7\alpha tH^2T_0^6/k^2 = (1/\eta^8)[3 - \tfrac{37}{7}\eta + \tfrac{13}{6}\eta^2] + \tfrac{5}{42} \qquad (30)$$

$$\eta = q/T_0 + 1 \qquad (31)$$

As can be seen in Fig. 2, Eq. (30) is in excellent agreement with both Goodman's solution[9] and the exact solution.[11]

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# Entrainment Theory for Compressible, Turbulent Boundary Layers on Adiabatic Walls

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## Nomenclature

$C_f$ = local skin-friction coefficient
$H$ = $\delta^*/\theta$

$$H_{1k} = \int_0^\delta \frac{u}{u_e}\,dy \Big/ \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy$$

$$H_k = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy \Big/ \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy$$

$H_1$ = $\Delta/\theta$

$$\bar{H} = \int_0^\delta \frac{\rho}{\rho_e}\left(1 - \frac{u}{u_e}\right) dy/\theta$$

$j$ = 0, 1 for two-dimensional and axisymmetrical flow, respectively
$M$ = Mach number
$r$ = recovery factor
$R$ = radius of the body of revolution
$T$ = static temperature
$T_r$ = recovery temperature
$T^*$ = Eckert's[15] reference temperature
$u,v$ = velocity components
$x,y$ = coordinates measured along and normal to the body surface
$\delta$ = boundary-layer thickness

$\delta^*$ = displacement thickness = $\int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$

$\Delta$ = $\delta - \delta^*$
$\mu^*$ = viscosity evaluated at $T^*$
$\rho$ = density

$\theta$ = momentum thickness = $\int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$

***Subscript***

$e$ = conditions at the outer edge of the boundary layer

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## Introduction

THERE are many situations where engineering designers are concerned only with the gross characteristics of boundary-layer flows and not the point-by-point variation of the flow variables. In such cases integral methods are often adopted due to their ease of implementation, speed of computation and relative accuracy. A particularly simple and accurate integral method and also one that is ideally suited for extension to more complicated situations is the entrainment theory of Head.[1] Although originally developed for two-dimensional, turbulent, incompressible boundary layers the entrainment theory has been successfully extended to include effects of three-dimensionality,[2] compressibility,[3–5] boundary-layer–inviscid flow interactions,[6] surface roughness,[7] and body rotation.[8,9] Standen[3] and So[4] based their extensions to compressible flow on different transformation theories that effectively reduce the compressible problem to an incompressible one whereas Green[5] developed a third transformation procedure as well as a direct method. Green found that the direct method compared much more favorably with his own data, taken downstream of a shock wave–boundary-layer interaction, than the transformation procedure. Based on available zero pressure gradient data, Green went on to present convincing arguments which cast serious doubts on the validity of existing transformations. Other authors[10,11] have also noted deficiencies in the existing transformations and Alber[10] indicated that certain transformations could lead to substantial errors at high Mach numbers. On the other hand, McDonald[11] suggested that an improved transformation might result from adopting a two-layer model for the compressible, turbulent boundary layer.

Based on the aforementioned information it appears that the most promising approach for developing an integral method for compressible, turbulent boundary layers is to consider a direct approach that is based on firm physical principles such as Green's[5] entrainment theory. The purpose of the present analysis is to establish a simplification of Green's direct compressible entrainment theory that is more in line with Head's[1] original theory for incompressible boundary layers. The results will be applicable to both two-dimensional and axisymmetrical flows.

## Analysis

The continuity equation for steady, compressible, turbulent flow may be written in the form[12]

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y + j[(\rho u/R)dR/dx] = 0$$

where the flow variables appear as time-averaged quantities. Integrating across the boundary layer gives

$$\frac{1}{\rho_e u_e}\frac{d}{dx}\left(\int_0^\delta \rho u dy\right) + \frac{j}{R}\frac{dR}{dx}\int_0^\delta \frac{\rho u}{\rho_e u_e} dy = \frac{1}{u_e}\left(u_e \frac{d\delta}{dx} - v_e\right) \quad (1)$$

where the right side is termed the dimensionless rate of entrainment, $F$. Expanding the left side of Eq. (1) and introducing the displacement thickness gives

$$\frac{d\Delta}{dx} = F + (M_e{}^2 - 1)\frac{\Delta}{u_e}\frac{du_e}{dx} - j\frac{\Delta}{R}\frac{dR}{dx} \quad (2)$$